

## Problem 12

In a water purification process, one- $n$ th of the impurity is removed in the first stage. In each succeeding stage, the amount of impurity removed is one- $n$ th of that removed in the preceding stage. Show that if  $n = 2$ , the water can be made as pure as you like, but that if  $n = 3$ , at least one-half of the impurity will remain no matter how many stages are used.

### Solution

Assuming there are an infinite number of purification stages, the percentage of impurity removed is given by

$$\left[ \overbrace{\left(\frac{1}{n}\right)}^{\text{stage 1}} + \overbrace{\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)}^{\text{stage 2}} + \overbrace{\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)}^{\text{stage 3}} + \cdots \right] \times 100\%$$

$$100\% \times \sum_{i=1}^{\infty} \left(\frac{1}{n}\right)^i$$

$$100\% \times \left[ -1 + \sum_{i=0}^{\infty} \left(\frac{1}{n}\right)^i \right]$$

$$100\% \times \left[ -1 + \frac{1}{1 - \left(\frac{1}{n}\right)} \right]$$

$$100\% \times \left( -1 + \frac{n}{n-1} \right)$$

$$100\% \times \left( \frac{1}{n-1} \right).$$

After an infinite number of purification stages with  $n = 2$  the percentage of impurity removed is

$$100\%.$$

After an infinite number of purification stages with  $n = 3$  the percentage of impurity removed is

$$50\%.$$